QUOTES AND NOTES ON LOGIC

I: The Paradoxes of Logic and Set Theory -- What is their SIGNIFICANCE?

(a) A Tapestry of Clues

1. Howard Eves, Introduction to the History of Mathematics, p. 351:

"The generalization to transfinite numbers and the abstract study of sets have widened and deepened many branches of mathematics, but, at the same time, they have revealed some very disturbing paradoxes which appear to lie in the innermost depths of mathematics. Here is where we seem to be today, and it may be that the second half of the twentieth century will witness the resolution of these critical problems."

2. Hermann Weyl, "Mathematics and Logic," American Mathematical Monthly, 53: 1946:

"We are less certain than ever about the ultimate foundations of (logic and) mathematics. Like everybody and everything in the world today, we have our 'crisis.' We have had it for nearly fifty years. Outwardly, it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively 'safe,' and has been a constant drain on the enthusiasm and determination with which I pursued my research work."

3. Bertrand Russell, Principia Mathematica, pp. 37; 61:

"An analysis of the paradoxes to be avoided shows that they all result from a certain kind of vicious circle.... In all the above contradictions (which are merely selections from an indefinite number) there is a common characteristic which we may describe as self-reference or reflexiveness."

4. Karl Marx, Economic-Philosophic Manuscripts of 1844, p. 204:

"Man is self-referring. His eye, his ear, etc., are self-referring; every one of his faculties has this quality of self-reference."

5. Bertrand Russell, Principia Mathematica, p. 38:

"...any limitation which makes it legitimate must make any statement about the totality fall outside the totality."

6. Alfred Schmidt, The Concept of Nature in Marx, p. 56:

"The reflective consciousness was for Marx simultaneously a moment of man's 'practical-critical' activity. The thought always enters into the reality mirrored by it as an essential moment."

7. Norbert Wiener, Cybernetics, p. 126:

"A proof represents a logical process which has come to a definite conclusion in a finite number of stages. However, a logical machine following definite rules may never come to a conclusion. It may go on grinding through different stages without ever coming to a stop, either by describing a pattern of activity of continually increasing complexity, or by going into a repetitive process like the end of a chess game in which there is a continuing cycle of perpetual check. This occurs in the case of some of the paradoxes of Cantor and Russell. Let us consider the class of all classes which are not members of themselves. Is this class a member of itself? If it is, it is certainly not a member of itself; and if it is not, it is equally certainly a member of itself. A machine to answer this question would give the successive temporary answers: 'yes,' 'no,' 'yes,' 'no,' and so on, and would never come to equilibrium."

8. Gregory Bateson, Steps to an Ecology of Mind, p. 281:

"The 'if... then...' of logic contains no time. But in the computer, cause and effect are used to simulate the 'if... then...' of logic; and all sequences of cause and effect necessarily involve time. (Conversely, we may say that in scientific explanations the 'if... then...' of logic is used to simulate the 'if... then...' of cause and effect.) The computer never truly encounters logical paradox, but only the simulation of paradox in trains of cause and effect. The computer therefore does not fade away. It merely oscillates."

9. NOTE: George Boole's 'Algebra of Thought' -- If the Russell-Whitehead project in Principia Mathematica aimed at reducing mathematics to formal logic (set theory), Boole's Laws of Thought exhibits the converse tendency to 'reduce logic to mathematics.' Boole models logic via an arithmetic and algebra restricted to the unit interval, 0 to 1. Boole's 'logical equations' are 'set equations.'

In this algebra the unknowns, variables, and numbers represent sets.* Thus, the numeral "1" (or 1/1) stands for "Universe," the set of all objects; "0" (or "0/1") stands for "Nothing," the empty set. The symbol 0/0 (since $0 \cdot x = 0$ for <u>any</u> set x) denotes the "indefinite" class ("some" objects), and "1/0" denotes the "Infinity" or the "impossible class" (since $0 \cdot x = 1$ holds for <u>no</u> set x). "Multiplication" models set <u>intersection</u>; thus if r = the set of red things and f = the set of flowers, then rf = fr = the set of <u>red flowers</u>. "Addition" models set <u>union</u> (<u>exclusive</u> 'or'). "Subtraction" models negation; "1-x" denotes the <u>complement</u> of the set x, i.e. the Universe EXCEPT x. Finally, "Division" models "abstraction"; i.e. if z = the set of red flowers = rf, then z/f = r = the set of all red things (or the quality "redness" in extension), thus abstracting "flowers" from "red flowers." Therefore the expression $xx = x^2$ represents <u>self</u>-intersection of the set x, and Boole's "fundamental law of thought" $x^2 = x^1$ is factorable into $x - x^2 = 0$ and thence into $x(1-x) = 0$ or "x times not-x is Nothing," which is just the Boolean version of the "law of non-contradiction": the intersection of a set and its negation is the empty set.

10. George Boole, <u>Laws of Thought</u>, pp. 49-51:

"That axiom of metaphysicians which is termed the principal of contradiction, and which affirms that it is impossible for any being to possess a quality, and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is $x^2 = x$... (1) .... I desire to direct attention also to the circumstance that the equation (1) in which that fundamental law of thought is expressed is an equation of the second degree.** Without speculating at all in this chapter upon the question, whether that circumstance is necessary in its own nature, we may venture to assert that if it had not existed, the whole procedure of the understanding would have been different from what it is. Thus it is a consequence of the fact that the fundamental equation of thought is of the second degree, that we perform the operation of analysis and classification, by division into pairs of opposites, or, as it is technically said, by <u>dichotomy</u>. Now if the equation in question had been of the third degree, still admitting of interpretation as such, the mental division must have been threefold in character, and we must have proceeded by a species of <u>trichotomy</u>, the real nature

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*[sets in turn represent ideas, concepts, "mental images" by way of the method of "extension." For example, the concept of the quality "redness" is represented by the <u>set</u> of <u>all</u> red things, the "<u>extension</u>" of the "intension": "redness." Hence Boole is aiming at a "caculus of concepts."]

**[i.e. is a <u>nonlinear</u> equation. Any equation $x^n = x$ where $n \neq 1$ would be a nonlinear equation.]

of which it is impossible for us, with our existing faculties, adequately to conceive, but the laws of which we might still investigate as an object of intellectual speculation... The law of thought expressed by equation (1) will, for reasons which are made apparent by the above discussion, be occasionally referred to as the "law of duality."

11. George Boole, Laws of Thought, p. 50n:

"Should it be here said that the existence of the equation $x^2 = x$ necessitates also the existence of the equation $x^3 = x$, which is of the third degree, and then inquired whether that equation does not indicate a process of trichotomy; the answer is, that the equation $x^3 = x$ is not interpretable in the system of logic. For writing it in either of the forms

$$x(1 - x)(1 + x) = 0, \qquad (2)$$

$$x(1 - x)(-1 - x) = 0, \qquad (3)$$

we see that its interpretation, if possible at all, must involve that of the factor $1 + x$, or of the factor $-1-x$. The former is not interpretable, because we cannot conceive of the addition of any class x to the universe 1; the latter is not interpretable, because the symbol -1 is not subject to the law $x(1-x) = 0$, to which all class symbols are subject. Hence the equation $x^3 = x$ admits of no interpretation analogous to that of the equation $x^2 = x$. Were the former equation, however, true independently of the latter, i.e. were the act of the mind which is denoted by the symbol x, such that its second repetition should reproduce the result of a single operation, but not its first or mere repetition, it is presumable that we should be able to interpret one of the forms (2), (3), which under the actual conditions of thought we cannot do. There exist operations, known to the mathematician, the law of which may be adequately expressed by the equation $x^3 = x$. But they are of a nature altogether foreign to the province of general reasoning."

12. Francesco Varela, "On Observing Natural Systems," Co-Evolution Quarterly, no. 10, Summer 1976, pp. 29-30:

"The fact is that, in Western thought, self-reference has been completely put aside... Now, an entirely new, fresh beginning was provided by G. Spencer Brown in his Laws of Form. Brown is an important key to what we're discussing, because he shows that the act of indication, of distinguishing something, is really what is happening at the basis of any description of any universe.

Distinguishing or not distinguishing, as he proves, is much more general, more fundamental, than true or false. True-False is just one particular case of the general act of distinguishing. Thus the notion of indication is the domain which he explores, the indication of forms, the most bare outline of anything we can describe. Brown strips systems down to their bare bones, which is just the pure indicational forms... Brown shows that, in the most precise mathematical sense, every description can be based, including logic of course, on the act of distinction. The calculus of these distinctions is what is contained in his book... It is from his grounds that a different view of self-reference can be developed. For, in its purest indicational form, self-reference appears as forms that in-form themselves. This may sound just poetic but it isn't. It also can be represented as very well-defined mathematical expression."

13. George Spencer Brown, Laws of Form, pp. viii - xi:

"Apart from the standard university logic problems, which the calculus published in this text renders so easy that we need not trouble ourselves further with them, perhaps the most significant thing, from the mathematical angle, that it enables us to do is to use complex values in the algebra of logic. They are the analogs, in ordinary algebra, to complex numbers $a+\sqrt{-1}\,b$. My brother and I had been using their Boolean counterparts in practical engineering work for several years before realizing what they were. Of course, being what they are, they work perfectly well, but understandably we felt a bit guilty about using them, just as the first mathematicians to use 'square roots of negative numbers' had felt guilty, because they too could see no plausible way of giving them a respectable academic meaning. All the same we were quite sure there was a perfectly good theory that would support them, if only we could think of it.

"The position is simply this. In ordinary algebra, complex values are accepted as a matter of course, and the more advanced techniques would be impossible without them. In Boolean algebra (and thus, for example, in all our reasoning processes) we disallow them. Whitehead and Russell introduced a special rule, which they called the Theory of Types, expressly to do so. Mistakenly, as it now turns out. So, in this field, the more advanced techniques, although not impossible, simply don't yet exist. At the present moment we are constrained, in our reasoning processes, to do it the way it was done in Aristotle's day. The poet Blake might have had some insight into this, for in 1788 he wrote that 'reason, or the ratio of all we have already known, is not the same that it will be when we know more.'

"Recalling Russell's connection with the Theory of Types, it was with some trepidation that I approached him in 1967 with the proof that it was unnecessary. To my relief he was delighted. The Theory was, he said, the most arbitrary thing he and Whitehead had ever had to do, not really a theory but a stop-gap, and he was glad to have lived long enough to see the matter resolved.

"Put as simply as I can make it, the resolution is as follows. All we have to show is that the self-referential paradoxes, discarded with the Theory of Types, are no worse than similar self-referential paradoxes, which are considered quite acceptable, in the ordinary theory of equations.

"The most famous such paradox in logic is in the statement, 'This statement is false.'

"Suppose we assume that a statement falls into one of three categories, true, false, or meaningless, and that a meaningful statement that is not true must be false, and one that is not false must be true. The statement under consideration does not appear to be meaningless (some philosophers have claimed that it is, but it is easy to refute this), so it must be true or false. If it is true, it must be, as it says, false. But if it is false, since this is what it says, it must be true.

"It has not hitherto been noticed that we have an equally vicious paradox in ordinary equation theory, because we have carefully guarded ourselves against expressing it in this way. Let us now do so.

"We will make assumptions quite analogous to those above. We assume that a number can be either positive, negative, or zero (corresponding to True, False, and Meaningless respectively, above). We assume further that a nonzero number that is not positive must be negative, and one that is not negative must be positive. We now consider the equation

$$x^2 + 1 = 0.$$

Transposing, we have

$$x^2 = -1$$

and dividing both sides by x gives

$$x = \frac{-1}{x} .$$

"We can see that this (like the analogous statement in logic) is self-referential; the root-value of x that we seek must be put back into the expression from which we seek it.

"Mere inspection shows that x must be a form of unity, or the equation would not balance numerically. We have assumed only two forms of unity, +1 and -1, so we may now try them each in turn. Set x = +1. This gives

$$+1 = \frac{-1}{+1} = -1$$

which is clearly paradoxical. So set x = -1. This time we have

$$-1 = \frac{-1}{-1} = +1$$

and it is equally paradoxical.

"Of course, as everybody knows, the paradox in this case is resolved by introducing a fourth class of number, called <u>imaginary</u>, so that we can say the roots of the equation above are $\pm$ i, where i is a new kind of unity that consists of a square root of minus one.

"What we do in Chapter 11 is extend the concept to Boolean algebras, which means that a valid argument may contain not just three classes of statement, but four: true, false, meaningless, and imaginary. The implications of this, in the fields of logic, philosophy, mathematics, and even physics, are profound.

"What is fascinating about the imaginary Boolean values once we admit them, is the light they apparently shed on our concepts of matter and time. It is, I guess, in the nature of us all to wonder why the universe appears just the way it does. Why, for example, does it not appear more symmetrical? Well, if you will be kind enough, and patient enough, to bear with me through the argument as it develops itself in this text, you will I think see, even though we begin it as symmetrically as we know how, that it becomes, of its own accord, less and less so as we proceed."

14. Francesco Varela, "A Calculus for Self-Reference," International Journal of General Systems, 2: 1975, p. 6:

"It would be of interest in itself to explore further the description of self-referential notions with the tools of higher degree* equations. Yet, as Spencer-Brown says, he has only indicated a direction for work, and not provided a firmly constructed theory of re-entering expressions (page xx)."

15. George Spencer-Brown, Laws of Form, pp. 99-100:

"The fact that imaginary values can be used to reason towards a real and certain answer, coupled with the fact that they are not so used in mathematical reasoning today, and also coupled with the fact that certain equations clearly cannot be solved without the use of imaginary values, means that there must be mathematical statements (whose truth or untruth is in fact perfectly decidable) which cannot be decided by the method of reasoning to which we have hitherto restricted ourselves.

"Generally speaking, if we confine our reasoning to an interpretation of Boolean equations of the first degree** only, we should expect to find theorems which will always defy decision, and the fact that we do seem to find such theorems in common arithmetic may serve, here, as a practical confirmation of this obvious prediction. To confirm it theoretically, we need only prove (1) that such theorems cannot be decided by reasoning of the first degree, and (2) that they can be decided by reasoning of a higher degree.†

16. Hermann Weyl, Philosophy of Mathematics and Natural Science, p. 50:

"...Here, at the farthest frontiers of set theory, actual contradictions did show up. But their root can only be seen in the boldness perpetrated from (the) beginning in mathematics, namely, of treating a field of constructive possibilities as a closed aggregate of objects existing in themselves."

17. Charles Musès, "The First Nondistributive Algebra," in E. R. Caianiello, editor, Functional Analysis and Optimization, 1966, pp. 209-211:

"The Greeks considered suspect and abnormal any number x such that $k \cdot x < 0$ where k was any positive number. Renaissance man, though he had long accepted

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*[nonlinear] **[linear Boolean equations]

†[i.e. by "nonlinear logic" expressed by "imaginary"-number-solvable Boolean nonlinear equations]

negative numbers as just as natural as positive numbers, still balked at x where $x^2 = -1$, although he used such numbers to solve some quadratic* equations. It took until the 19th century until man's mind could regard these numbers too as nonpathological, although the designation "imaginary" still clings to them. In the 20th century, Eduard Study first considered a number x not equal to zero and such that $x^2 = 0$; although Study still had no realization that this implies also $x^0 = 0$, and an advanced form of nondistributive multiplication....

"The higher kinds of number for the first time yield the concrete hope of placing the profound and subtle characteristics of bio-, psycho-, and socio-transformations and processes on an adequate mathematical basis. Such kinds of number would thus introduce the humanities to their appropriate mathematics, which will not do them the grave and unscientific injustice of forcing them to fit some Procrustean bed of inadequate hypothesis or reductive definition. Man and man's sciences are now ready to go beyond the square root of minus one. With each new and higher kind of number a new and deeper algebra and arithmetic become possible, and hence a new and deeper functional analysis."

(b) NOTE: On Dr. Musès' "Hypernumbers" and "Boolean Equations of Higher Degree" --

The "hypernumbers" proposed by Dr. Musès, particularly those characterizable as "proper" nth roots of cardinal one, ordinary unity, represent themselves new kinds or "qualities" of unity, other than +1, -1, and i. They would give new solutions, not only to the Boolean "uninterpretable" set-equation $x^3 = x$ -- [which may be considered a 'paradox' in that it is reflexive (self-intersection), in that it differs from the law of noncontradiction, contained uniquely in $x^2 = x$, and in that it is "self-referential" or "impredicative," i.e. falls under the form '$x \equiv f(x)$'] -- but to even higher degree Boolean equations, such as $x^5 = x$ and $x^7 = x$ as well.

By "proper" root Dr. Musès means a root other than the same number again from which the root itself was extracted (just as set X is said to be an improper sub-set of itself). +1 itself is always the improper nth root of +1.

*[second-degree]

Dr. Musès defines a hypernumber ε, a proper square root of <u>positive</u> unity, such that $\varepsilon^3 = \varepsilon$, because $\varepsilon^2 = +1$; thus a root, together with 0, +1 and -1, of Boole's $x^3 = x$. The old familiar i itself is a root of $x^5 = x$. Dr. Musès also proposes a number w such that $w = \sqrt[6]{+1}$, so that $w = \sqrt[7]{w} = w^7$, thus solving the Boolean $x^7 = x$. Musès' p, such that $p \neq 0$, but $p^2 = 0$, i.e. $p = \sqrt[2]{0}$ and $p^9 = p$, solves $x^9 = x$.

Since, in Boole's algebra, the ordinary kind of unity, 1, refers to the Universal Set or "Universe," we might wonder what kinds of 'Universes' and 'Sets' <u>these</u> other kinds of unity might represent!

Noting that "equations of higher degree" (of degree >1) are also known as "<u>nonlinear</u> equations," we might characterize the logics described by these 'Boolean equations of higher degree' as '<u>nonlinear logics</u>,' and ponder what such logics might be about, and what relations they might have to ordinary (formal or Aristotelian) '<u>linear</u>' logic. We might also wonder, what kind of <u>mathematics</u> would grow up upon the foundation of such "non-formal" logics, how it might differ from the mathematics we know, and whether with it we might be able to solve problems which, in the mathematics grown from formal logic, appear "paradoxical" or "insoluble."

(c) <u>Suggested Readings</u>

1. Ernest Nagel, "Impossible Numbers": A Chapter in the History of Modern Logic, in <u>Studies in the History of Ideas</u>, volume 3, 1935. (UC B21/C7 Main)

2. Sir William Rowan Hamilton, "Algebra as the Science of Pure Time," <u>Transactions</u>, Royal Irish Academy, Vol. XVII, 1837.